Renormalization Induced Quantum Small-world Networks

wbravo@mail.ustc.edu.cn
Outline

Background and Motivation

Basic Rules for Quantum Networks

Methodology: Coupled Renormalization

Proofs: Fractal to Small-World Networks

Comparison With Entanglement Percolation

Summary and Outlook
Quick Review of Entanglement

- Nonseparable: $\hat{\rho}_{AB} \neq \hat{\rho}_A \otimes \hat{\rho}_B$

- Nonlocal correlation VS local realism
  Lies at the heart of quantum physics

- As real as energy – a novel kind of resource

- Maximally or Partially Entanglement
  $|\psi\rangle = \alpha |01\rangle + \beta |10\rangle$

- Fidelity: $F = \langle \psi | \hat{\rho} | \psi \rangle$
Preparation of entanglement

\[ |\Phi^{\pm}_{12} \rangle = \frac{1}{\sqrt{2}} \left( |H\rangle_1 |H\rangle_2 \pm |V\rangle_1 |V\rangle_2 \right) \]

\[ |\Psi^{\pm}_{12} \rangle = \frac{1}{\sqrt{2}} \left( |H\rangle_1 |V\rangle_2 \pm |V\rangle_1 |H\rangle_2 \right) \]

spontaneous parametric down-conversion SPDC

Entanglement—assisted quantum communication

Teleportation
- Teleportation
- Quantum cryptography (Ekert 91)
- Dense coding
- Others: BB84, B92

Entanglement Swapping
Toward long distance quantum communication

**Problem:** photon loss and state decoherence

*Exponentially* decay of fidelity with distance
Enlightenments from Complex Networks

Table: General features

- Scale-free: $p(k) \sim k^{-\gamma}$
- Small-world: $\bar{\ell} \sim \ln N$
- Self-similarity: $N_B / N \sim \ell_B^{-d_B}$
- Modularity

Open problem:

- What’s the possible topological architecture of quantum networks?
- Are quantum networks exceptions?
Relationship with classical communication networks

- **Constraints:**
  Local attachment

- **Assumption:**
  Scale-free-fractal network observed at a length-scale

- Why is it reasonable?
  Two facts:

  - Classical communication is necessary. To some extent, quantum networks are embed in classical communication networks.

  - Some *organizing principle* that shapes the topology is required to control the allocation and layout of long-range entangled links.
Brief introduction to renormalization

\[
\frac{N_B}{N} \sim \ell_B^{-d_B}, \quad k' \sim \ell_B^{-d_k}k, \quad \gamma = 1 + d_B/d_k
\]

maximum-excluded-mass-burning

Figure 10. Illustration of the MEMB algorithm for \( r_B = 1 \). Upper row: Calculation of the box centres. (a) We calculate the excluded mass for each node. (b) The node with maximum mass becomes a centre and the excluded masses are recalculated. (c) A new centre is chosen. Now, the entire network is covered with these two centres. Bottom row: calculation of the boxes. (d) Each box includes initially only the centre. Starting from the centres we calculate the distance of each network node to the closest centre. (e) We assign each node to its nearest box.
Renormalization and its relationship with quantum repeaters

(a) A C₁ C₂ B

(b) $\ell_c = 3$

(c) Scale-Free-Fractal Network

(d) Coupled Network
Map distribution of shortcuts to coupled renormalization

**Corresponding relationships**

- **Nesting level**

\[ n_c \sim \frac{\ln N}{d_B \ln \ell_c}. \]

- **hierarchy**

- **Transforming length-scale**

**Highlights**

For detailed explanations. The highlights of this scenario are as follows. The distribution of shortcuts becomes a process of collectively implementing QRP across the entire network, where the hierarchy of measurement is preserved at network level, while the scale-free-fractal structure is kept at all length-scales. And large-scale scale-free-fractal QRN are built via consecutive enlargement with some length-scales. Besides, we will both analytically and numerically prove that small-world is obtained as well.
Minimal Model

Parameters: $n, s, a, e$

\[
\tilde{N}(t) = n\tilde{N}(t-1),
\]

\[
\tilde{k}(t) = s\tilde{k}(t-1),
\]

\[
\tilde{L}(t) + L_0 = a(\tilde{L}(t-1) + L_0),
\]

\[
d_B = \ln(2m + 1)/\ln 3,
\]

\[
\gamma = 1 + \ln(2m + 1)/\ln(m + e).
\]

$m = 2, e = 0$, then $d_B \approx 1.46.$
Hierarchical rooting method

\[
\frac{N_B}{N} \sim \ell_B^{d_B}. \quad D_0 \sim N^{\frac{1}{d_B}}, \quad D_B(\ell_c) \sim D_0/\ell_c.
\]

\[
D_C(\ell_c) \approx \frac{D_0}{\ell_c} + \ell_c - 1.
\]

\[
\ell \sim N^{\gamma - 2/\gamma - 1}. \quad \bar{\ell} \sim D_0^d, \quad d = d_B - d_k.
\]

\[
\bar{\ell}_s(\ell_c) \approx \left(\frac{D_0}{\ell_c}\right)^d + \ell_c^d.
\]

\[
\ell_o = \sqrt{D_0}, \quad D_{\text{min}}(N) = 2\sqrt{D_0} - 1,
\]

\[
\bar{\ell}_{\text{min}}(N) = 2D_0^{0.5d}, \quad \bar{\ell}_{\text{min}}(N) = D_{\text{min}}(N)/2. = \sqrt{D_0} = N^{\frac{1}{2d_B}}.
\]
Proofs(1)—definition

\[ D(n, \ell_c) \approx \frac{D_0}{\ell_c^m} + n(\ell_c - 1), \ell_c^{n_c} \sim D_0 \]

\[ D(n_c, \ell_c) \approx n_c(\ell_c - 1) \]

\[ n_c \sim \frac{\ln N}{d_B \ln \ell_c}, \quad D(n_c, \ell_c) \approx \frac{\ell_c - 1}{d_B \ln \ell_c} \ln N. \]
Statistical properties of coupled networks

Self-similarity is closely associated with length-scale

Gaussian Distribution
Small-World to Fractal Transition in Complex Networks: A Renormalization Group Approach

Hernán D. Rozenfeld, Chaoming Song, and Hernán A. Makse
Proofs(2)—renormalization flow

\[ p(\ell) \sim \ell^{-d_B} / \ell_B^{d_B-1} = \ell^{-(2d_B-1)}, \]

\[ s = 2 - 1/d_B, \lambda = 2 - s = 1/d_B \approx 0.68. \]

\[ f_B - f_0 = \sum_{j=m}^{n} 2 \ell_{c}^{-jd_B} \ell_B^{d_B} \sim (f' - f_0) \xi_B^{1-\beta}, \]

\[ 1 - \beta \approx 0.89 \]

\[ f_B - f_0 = 2 \ell_{c}^{-d_B} N/N_B = 2(\ell_B/\ell_c)^{d_B}. \]

\[ f_B - f_0 \to 0, \lambda \ll 0. \]

\[ \ell_{\min}(N) = \sqrt{D_0} = N^{\frac{1}{2d_B}}, \]

Conclusion

Iterative CR:

Single CR:
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Entanglement percolation

Figure 3 Example of a quantum network where entanglement percolation and CEP are not equivalent. Each node is connected by a state consisting of two copies

\[ |\psi\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle, \]
\[ \phi_1 = \min[1, 2(1 - \lambda_0)]. \]

Q-swap

\[ P(k) = Ck^{-\tau}e^{-k/\kappa} \]
Simulation test in Internet

**Drawback:** the size of spanning cluster decreases rapidly

**Personal comments**

- *Optimal strategy is not optimal* compared with classical entanglement percolation in some important cases.

- **Attractive case:**
  real scale-free networks $2 < \gamma < 3$

- *Absence of percolation threshold:* $N \to \infty$, $p_c \to 0$

- **Disadvantage:** *poor scalability* in space.

- Unable to replace quantum repeater networks.
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Summary and outlook

- Constraints and assumption

- Iterative renormalization is able to trigger fractal to small-world transition

- Advantage of our scenario

- Shift of perspective caused by the crossover study
END!

Thank you!