Growing directed networks: organization and dynamics

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Abstract. We study the organization and dynamics of growing directed networks. These networks are built by adding nodes successively in such a way that each new node has K directed links to the existing ones. The organization of a growing directed network is analyzed in terms of the number of 'descendants' of each node in the network. We show that the distribution $P(S)$ of the size, S, of the descendant cluster is described generically by a power-law, $P(S) \sim S^{-\eta}$, where the exponent η depends on the value of K as well as the strength of preferential attachment. We determine that, in the case of growing random directed networks without any preferential attachment, η is given by $1 + 1/K$. We also show that the Boolean dynamics of these networks is stable for any value of K. However, with a small fraction of reversal in the direction of the links, the dynamics of growing directed networks appears to operate on 'the edge of chaos' with a power-law distribution of the cycle lengths. We suggest that the growing directed network may serve as another paradigm for the emergence of the scale-free features in network organization and dynamics.

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The dynamics of complex adaptive systems is strongly influenced by the way the elements of the network are connected and the way these elements interact. The organization of real-world networks [1]–[3] has attracted intensive interest following the seminal work of Barabási and Albert [4] on scale-free networks. The degree distribution $P(k)$, which gives the probability that a randomly selected node has exactly k edges, has been used as the most important characterization of complex networks. Power-law or scale-free degree distributions have been found in many real-world complex networks, such as the Internet, cellular metabolic networks, research collaboration network, and the World Wide Web [4, 5]. Most studies of the network have been focused on network topology, but there are also a few studies of dynamical processes on these networks [6]–[8]. In particular, Aldana and Cluzel demonstrated that the scale-free topology of the network favors robust dynamics [9]. Nevertheless, much work is still needed to characterize and classify network dynamics and organization.

In this paper, we study a generic class of growing directed networks, which are grown by adding nodes successively, just as in the well-known Barabási–Albert model. But we consider the resulting network only as a directed network, and we focus on the limiting case of a citation-like network in which the new node is directed to the existing nodes and not the other way around. Except for the initial cluster of $K + 1$ nodes, which are linked in both directions, all other nodes added have K directed links to the existing ones. In terms of dynamics the existing nodes are not controlled by the new node added, thus the network can be viewed essentially as a hierarchical feed-forward network. To make our discussion relevant to many real world networks which typically have a fraction of feedback links, we also consider the modified network that contains a small fraction of link reversals.

We investigate the global organization of these growing directed networks. Unlike the undirected version of the network, the influence of a node on the others in the directed network may be limited. For each node we can define a descendant cluster consisting of all the nodes that are linked to it directly or indirectly through intermediate nodes. More precisely, the descendant cluster of node v is the set of all nodes from which node v can be reached by following a path of directed links. This is the same as the in-component defined in [10]. The possible impact of a given node on the others can be characterized by the size of its descendant cluster. The cluster size distribution gives an overall description of the network organization in terms of the potential influence of one node on the others. It is a better measure of the global organization of the directed network than the degree distribution, which is essentially a measure of local connectivity in the network. We show that, as far as the cluster size distribution is concerned, growing directed networks are generically scale-free, irrespective of the strength of preferential attachment and the value of K.

We also investigate the Boolean dynamics of these growing networks. The study of the dynamics of Boolean networks was pioneered by Kauffman [11, 12], who focused primarily on random directed networks. Kauffman's NK model (Kauffman net) consists of N nodes and K directed links per node such that each node is controlled by K other nodes. The model has been used as a prototypical model of gene regulation and control. Kauffman suggests that gene networks operate on the 'edge of chaos' (in a critical phase), as evolution demands that there be sensitivity to perturbations and mutations. In Kauffman's NK model, the critical phase (defined as the edge of chaos) occurs only at the specific parameter value $K = 2$ [11, 13]. For $K > 2$ the dynamics is chaotic. We show that, in contrast, the dynamics of growing directed networks (with a small fraction of link reversals) appears to operate on the 'edge of chaos' for a wide range of values of K.
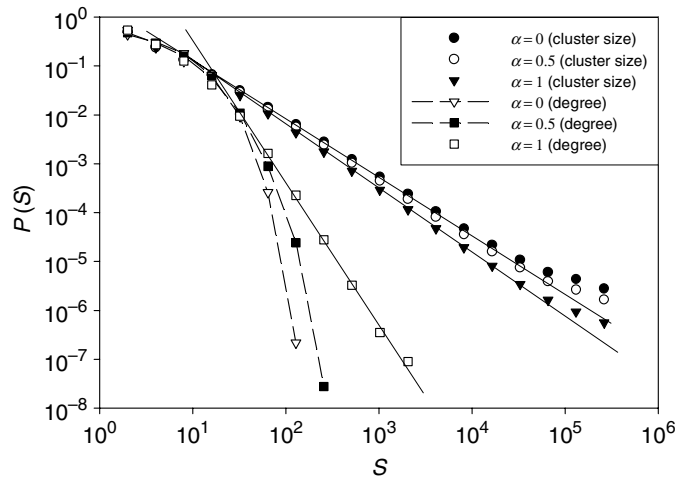
Figure 1. Model A: the descendant cluster size distribution for $\alpha = 0$, 0.5 and 1. $N = 200\,000$ and $K = 5$ are used.

Two models of growing directed networks are considered in our study. In model A, the network is grown in the same way as in the undirected version [14]. Krapivsky–Redner–Leyvraz (KRL) network model is built by adding sites that link to earlier sites with a probability depending on the number of pre-existing links k to that site. They found that for homogeneous connection kernels, $P_k \sim k^{-\alpha}$, different behaviors arise for $\alpha < 1$, $\alpha > 1$ and $\alpha = 1$. We start with an initial cluster of $K + 1$ nodes, which are fully connected (two directed links between each pairs of nodes). At each stage, we add a new node with K links to K of the nodes already present in the network. The link is directed from the new node to an existing node, meaning that the new node is a descendant of the existing one. We assume that the probability of connecting a new node to an existing one with degree k is proportional to k^α, where k is the total degree of the node: $k = k_{\mathrm{in}} + k_{\mathrm{out}}$ (k_{in} is the number of incoming links and k_{out} is the number of outgoing links; for this model $k_{\mathrm{out}} \equiv K$). For $\alpha > 0$, we have preferential attachment. The undirected version of the network with $\alpha = 1$ corresponds to the Barabási–Albert model. The power-law degree distribution can only be observed for $\alpha = 1$ [14, 15]. As far as the degree distribution is concerned, the network is only scale-free when $\alpha = 1$. In contrast, we found that the power-law distribution is in fact quite generic in the size distribution of descendant clusters. Figure 1 shows such size distribution for $\alpha = 0$, 0.5, 1.0 and $K = 5$. The degree distributions are also shown in the figure for comparison. For $\alpha \leqslant 1.0$ the cluster size distribution can be described very well by a power law $P(S) \sim S^{-\eta}$. (The deviation is noticeable only when $S \sim O(N)$.) For $\alpha = 0$ and 0.5, we found the exponent $\eta \approx 1.2$, and for $\alpha = 1$, $\eta \approx 1.3$. We have checked that the power laws exist for a wide range of values of K; the exponent depends on both K and α. Even though $\alpha = 1$ is a special case for the degree distribution (as can also be seen from figure 1), it is not for the descendant cluster distribution. The power-law distribution of descendant cluster sizes is rather generic in growing directed networks.

In the mean time, we can see all these power laws follow the finite size scaling. In figure 2, it is shown that all the data for $\alpha = 0$. $N = 50\,000$, $100\,000$, $200\,000$ and $K = 5$ fall on one line with $\eta \approx 1.2$.

We also consider a variant of model A, which we refer to as model B. In this model, we choose the growth rule such that the probability of connecting to the node with in-degree k_{in} is
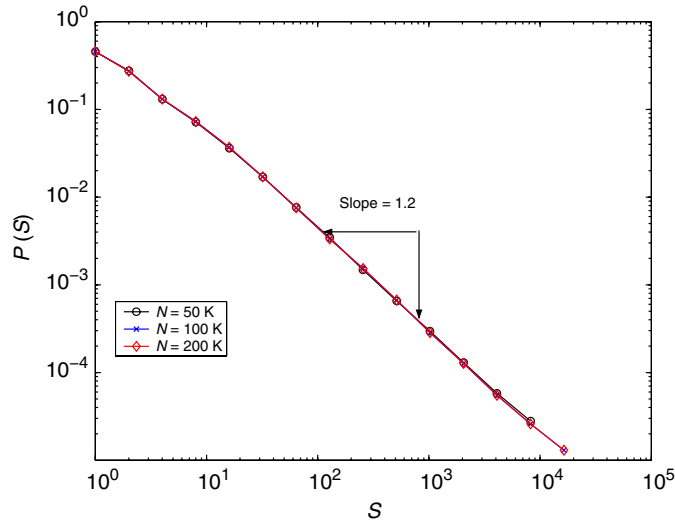
Figure 2. Model A: the descendant cluster size distribution for $\alpha = 0$. $N = 50\,000, 100\,000, 200\,000$ and $K = 5$ are used.
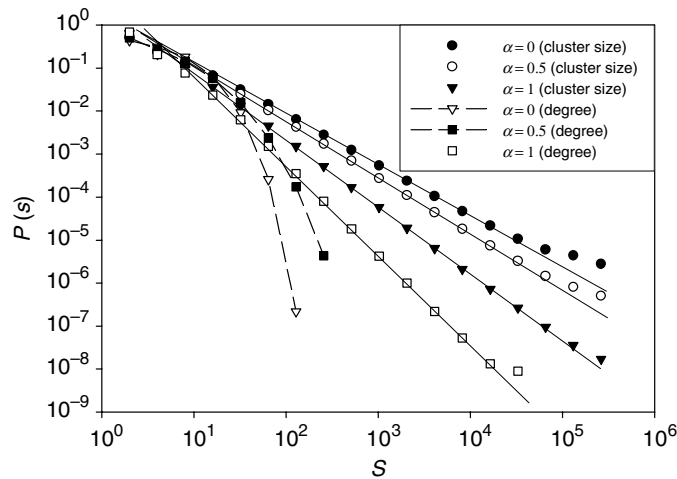

Figure 3. Model B: the descendant cluster size distribution for $\alpha = 0, 0.5$ and 1. $N = 200\,000$ and $K = 5$ are used.

proportional to $k_{\text{in}}^{\alpha} + 1$. The constant 1 is added to give a nonzero starting weight to the nodes that have not been connected to. Again we obtain power-law cluster size distributions, which are plotted in figure 3. The exponents obtained for the cluster distribution are different from those of model A. We obtained $\eta \approx 1.2$ for $\alpha = 0$, $\eta \approx 1.3$ for $\alpha = 0.5$, and $\eta \approx 1.6$ for $\alpha = 1$.

Models A and B are the same model when $\alpha = 0$ (no preferential attachment). For this case we can write down a master equation for the cluster size distribution. Let $n(N, S)$ be the number of clusters of size S when N nodes are present. Now we add a new node to the network. For $N \gg K$, $n(N, S)$ evolves according to the following equation:

$$n(N+1, S+1) = n(N, S) \left[1 - \left(1 - \frac{S}{N} \right)^K \right] + n(N, S+1) \left(1 - \frac{S+1}{N} \right)^K, \quad (1)$$
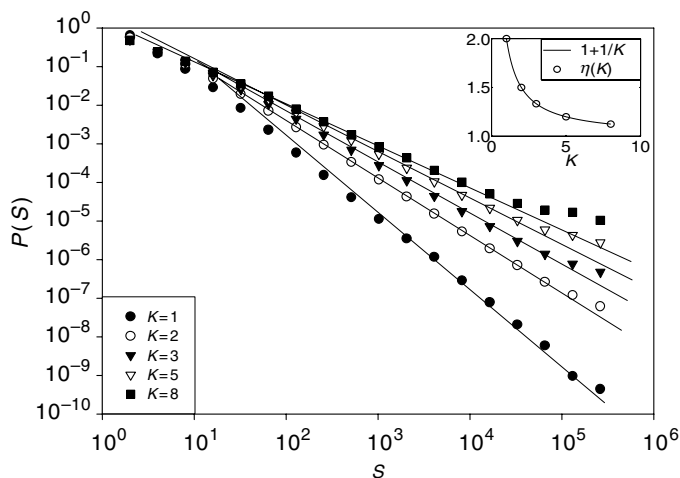
Figure 4. The cluster size distribution for $\alpha = 0$, $K = 1, 2, 3, 5, 8$ and $N = 200\,000$. The slopes of the lines drawn can be given by the value $(-\frac{1+K}{K})$.

where $(1 - \frac{S}{N})^K$ is the probability that the new node added does not link to any of the S nodes in a given cluster of size S. For $1 \ll S \ll N$, $n(N, S)$ can be approximated as $n(N, S) = N * p(s)$, where $s = S/N$ and $p(s)$ is the probability density function for the size distribution. In addition, $1 - (1 - \frac{S}{N})^K \approx KS/N = Ks$. In terms of $p(s)$ the above equation can be rewritten as

$$p\left(s + \frac{1}{N}\right) = -N\left[K\left(s + \frac{1}{N}\right)p\left(s + \frac{1}{N}\right) - Ksp(s)\right]. \tag{2}$$

Neglecting the terms of order $1/N$ and higher, we have

$$Ks\frac{\mathrm{d}p}{\mathrm{d}s} = -(K + 1)p(s). \tag{3}$$

This leads to $p(s) \propto s^{-\eta}$ with $\eta = 1 + 1/K$. Figure 4 shows the cluster size distribution for $\alpha = 0$ and $K = 1, 2, 3, 5$ and 8. The inset in figure 4 shows clearly that the numerical values of the exponent η for different K agree very well with the analytical values.

In figures 1, 3 and 4, we may see that all the curves bend upwards for large size s. This can be understood as a finite size effect. When the size approaches that of the whole network, the big size cluster appears with a probability appreciably larger than the value predicted by the power law scaling.

The cluster distribution for the special case of $\alpha = 0$ and $K = 1$ was first obtained by Krapivsky and Redner [10]. They obtained $\eta = 2$ in agreement with our analysis. They also pointed out that the distribution for the special case of $\alpha = 1$ and $K = 1$ can also be obtained analytically, and η is again equal to 2. This is not surprising, for both cases, the probability that a new node is added to a cluster of size S is simply S/N. This can be seen from the fact that, for $\alpha = 1$, this probability is proportional to $\sum k_{\mathrm{in}}/N$, which is the same as S/N, because $\sum k_{\mathrm{in}} = \sum k_{\mathrm{out}} = S$ for the cluster. Thus exactly the same master equation applies for both cases. We have also checked, numerically, that η is in fact equal to 2 for $0 \leqslant \alpha \leqslant 1$.

We can also generalize the model to the case $K < 1$. The meaning of K for this case is the probability that the new node gets connected to a randomly chosen existing node (the resulting network consists of disconnected components). Let us consider again the special case of $\alpha = 0$.

It is easy to check that the same master equation (equation (2)) can be used to determine $p(s)$, leading again to $p(s) \propto s^{-\eta}$ with $\eta = 1 + 1/K > 2$. It is interesting to note the component size distribution of growing (undirected) networks in the subcritical regime follows exactly the same scaling (see equation (31) of [16], which contains a review of universal properties of growing networks and earlier references on this subject). This is not surprising as the same master equation applies to both the distribution of the descendant cluster and that of the component size. However, there is no direct one-to-one mapping which maps a descendant cluster to a component of the network (each node has a descendant cluster, but the number of connected components is typically much less than the number of nodes).

We now turn to the Boolean dynamics of growing directed networks. The study of the dynamics of Boolean networks was pioneered by Kauffman, who used it as a prototypical model of a genetic regulatory network. In a Boolean network, each node is represented by an on–off switch which is a function of the binary output from some other nodes. In the Kauffman net, each node is controlled by K other nodes chosen randomly. The dynamics of the Kauffman net depends crucially on K. For $K = 1$ the dynamics converges to a fixed point or a limit cycle (this is the ordered phase). For $K = 2$, the system is at the 'edge of chaos' where cycles of many different lengths can appear. For $K > 2$, the system is in the chaotic phase.

The Boolean dynamics of our growing directed network is always in the ordered phase with the maximum period equal to 2^{K+1}. This is due to the fact that, by construction, the initial $K + 1$ nodes are mutually connected; they are not controlled by the nodes added to the network later. Thus the maximum period for the dynamics of this initial cluster is 2^{K+1}. As the new nodes are controlled only by the existing nodes in the cluster, it is easy to show, by induction, that the period of the dynamics of the entire system is the same as the period of the dynamics of the initial cluster. Thus the dynamics of the growing directed network is in the ordered phase irrespective of the values of K and α.

Many real-world networks, with the possible exception of citation-like networks, are not simple feed-forward directed networks we considered above. There typically exist a certain fraction of feedback links (the new node controls the existing one). To model these networks, we start with the original feed-forward network. Then, with a probability $q = p/K$ (here p is the probability that a node has a feedback link), we reverse the direction of the links. Here we focus on the case of $\alpha = 0$ (growing random directed networks). For sufficiently small values of p, the Boolean dynamics of these modified networks is similar to that of the Kauffman net when $K = 2$ [17]: depending on the initial conditions, cycles of a wide range of lengths appear. We performed extensive simulation of the Boolean dynamics of these networks for various values of α, K and N. The length of the simulation is up to 10^7 time steps so that we can detect the period of the dynamics up to the order of 10^7. The statistical results were obtained by averaging over 2000–200 000 independent runs. Figure 5 shows the distribution of the cycle length (period), which can be described as a power law $P(T) \sim T^{-\theta}$, where the exponent increases as p increases. We also found that there is a threshold value p_c, which depends on both K and N. The power law distribution of $P(T)$ occurs for $p < p_c$, with the exponent θ approaching 1 as $p \to p_c$. For $p > p_c$, the fraction of chaotic dynamics (numerically we classify the dynamics as chaotic if the period is greater than 10^7) increases rapidly. There is a clear transition from the 'edge of chaos' regime to a chaotic regime. Figure 6 shows the fraction of chaotic dynamics as a function of p, for $K = 4$ and 6, $N = 201$, 401 and 801. The transition is rather sharp for large N. We have also checked that the same qualitative behaviors can be observed for small α ($\alpha < 0.5$). However, for α close to 1, even though there is still a
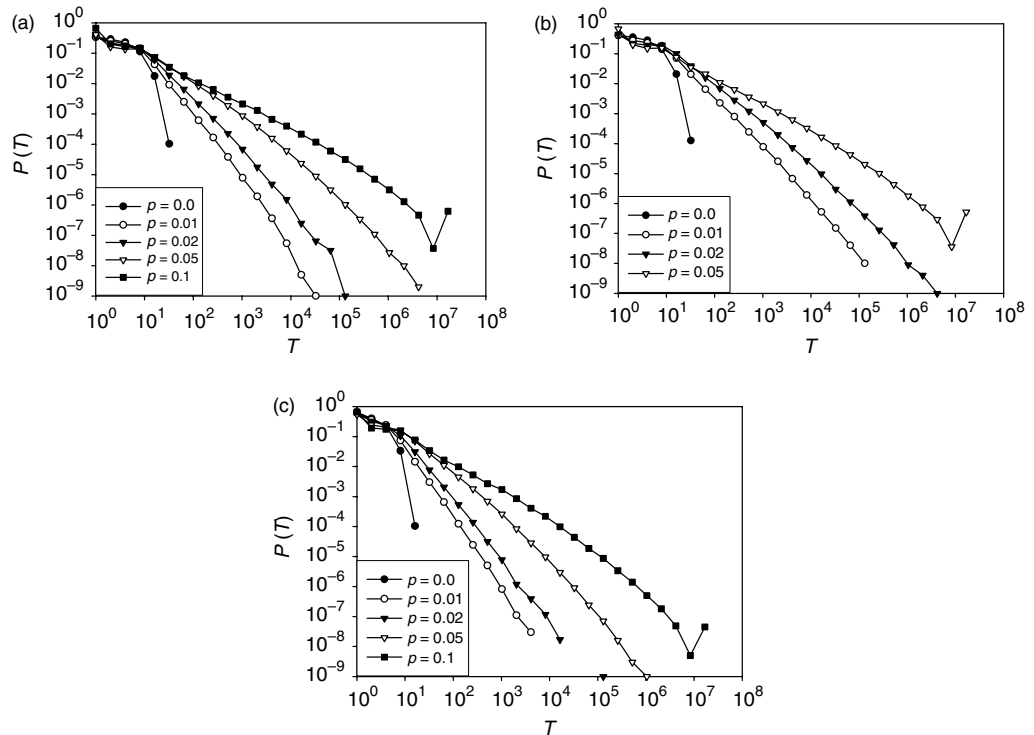
Figure 5. The distribution of cycle lengths in growing random directed networks ($\alpha = 0$) for the cases: (a) $K = 6$, $N = 401$, (b) $K = 6$, $N = 801$, and (c) $K = 4$, $N = 801$. The distribution (for $0 < p < p_c$) can be approximated by a power law $P(T) \sim T^{-\theta}$.
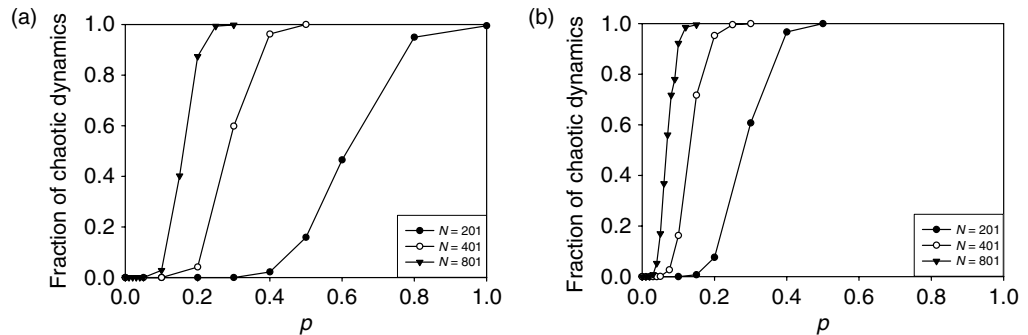

Figure 6. The fraction of chaotic dynamics found in growing random directed networks as a function of p for $N = 201, 401, 801$, and (a) $K = 4$ and (b) $K = 6$.

transition to a chaotic regime, there is no 'edge of chaos' regime for $p < p_c$ (the distribution $P(T)$ is broad, but it cannot be described using a power law). This shows that the dynamical properties of the network are not necessarily correlated with the local degree distribution. The limit of large N can also be seen from figure 6 where when N approaches infinity, all finite p values will lead to almost the whole case belonging to chaotic dynamics. However, it remains a challenging task to identify the key features of the global organization of the networks that directly affect the dynamics [18]–[21].

In conclusion, we have studied the organization and Boolean dynamics of growing directed networks. In terms of clusters of descendants, the size distribution exhibits a robust power law for a wide range of K and α values. The Boolean dynamics of the networks is very stable with the maximum period equal to 2^{K+1}. However, with a small fraction of link reversals, the dynamics appears to operate on the 'edge of chaos' with a power-law distribution of the cycle lengths. This critical regime is rather generic and can be obtained without a fine tuning of the parameters K and α, in contrast to the original Kauffman model. With its generic scale-free features in the organization and dynamics, the growing directed network serves as another paradigm for the emergence of scale-free dynamical and organizational properties as exhibited in many real-world networks.

Acknowledgments

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