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Analytical studies on a modified Nagel–Schreckenberg model with the Fukui–Ishibashi acceleration rule

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Abstract

We propose and study a one-dimensional traffic flow cellular automaton model of high-speed vehicles with the Fukui–Ishibashi-type (FI) acceleration rule for all cars, and the Nagel–Schreckenberg-type (NS) stochastic delay mechanism. We obtain analytically the fundamental diagrams of the average speed and vehicle flux depending on the vehicle density and stochastic delay probability. Our theoretical results are in excellent agreement with numerical simulations.

1. Introduction

Traffic flow that displays various complex behaviors is a kind of many-body systems of strongly interacting vehicles. One of the approaches to microscopic traffic processes is based on cellular automaton (CA) [1], which treats the motions of cars as hopping processes on one-dimensional lattices. In general, CA are idealization of physical systems in which both space and time are assumed to be discrete and each of the interacting units can have only a finite number of discrete states. In the past few decades, CA models for traffic flow have attracted much interest of a community of physicists [1,2]. Compared with other dynamical approaches, for instance the car-following theory approach, CA models are conceptually simpler, and can be easily implemented on computers for numerical investigations [3].

In the CA models of traffic the position, speed, acceleration as well as time are treated as discrete variables. In this approach, a lane is represented by a one-dimension lattice. Each of the lattice sites represents a "cell" which can be either empty or occupied by at most one "vehicle" at a given instant of time. At each discrete time step $t \rightarrow t+1$ the state of the system is updated following a well defined rule. Two popular one-dimensional (1D) CA models are the Nagel–Schreckenberg (NS) model [4] and the Fukui–Ishibashi (FI) model [5], where periodic boundary conditions are used to mimic the traffic flow on highway.

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In this paper we consider every car can accelerate to $v_{\max}$ during one step if there are $v_{\max}$ or more empty sites in front of the car, i.e. the acceleration rule of FI. Although the model is less realistic than the NS model, it is interesting due to its simplicity and leads to considerable improvement of the flow for higher velocities [6].

This paper is organized as follows. In Section 2, we give the definition of the model and the evolution equations for the inter-car spacings. Next, we present the fundamental diagrams of the average speed and vehicle flux depending on the vehicle density and stochastic delay probability, compared with the numerical simulations. In Section 4, we summary with a discussion of our method.

2. The model and analytical solution of asymptotic average velocity

The modified Nagel–Schreckenberg model with the Fukui–Ishibashi acceleration rule is a probabilistic cell automaton, in which space and time are discrete and hence also the velocities. The road of length L is divided into cells of certain length, each of which can either be empty or occupied by just one car. The state of the nth car ($n = 1, \ldots, N$) is characterized by the momentary velocity $v_n(t)$. In this model, the maximum speed is fixed as $v_{\max} = M$, and the vehicle density is $\rho = N/L$. Let $C_n(t)$ represent the number of empty sites in front of the nth vehicle at the time step t, then we have

$$C_n(t+1) = C_n(t) + v_{n+1}(t) - v_n(t). \tag{1}$$

As a function of the inter-car spacing $C_n(t)$ and the stochastic delay probability f, the velocity of the nth car at time step t can be written as:

$$v_n(t) = F_M(f, C_n(t)), \tag{2}$$

where

$$F_M(f, C) = \begin{cases} 0, & \text{if } C = 0 \\ C - 1 & \text{with probability } f, \text{if } 0 < C < M \\ C & \text{with probability } 1 - f, \text{if } 0 < C < M \\ M - 1 & \text{with probability } f, \text{if } C \geqslant M \\ M & \text{with probability } 1 - f, \text{if } C \geqslant M. \end{cases} \tag{3}$$

Here, as one can see, we have approximately adopted the acceleration rule introduced by Fukui and Ishibashi. The FI model can be considered as a NS model for "aggressive driving", since the rules of the FI are nearly identical to the NS model, except that the acceleration rule has been changed from "the vehicle speed is at most increased by 1 at each step" to "every car accelerates to $v_{\max}$" (which is adopted in the present model) and the stochastic delay mechanism has been modified as "only cars with $v_{\max}$ will delay stochastically" (which is not adopted here). For $v_{\max} = 1$, this may not change anything, however, for higher velocities it will lead to a different result.

If $N_i(t)$ represents the number of inter-car spacings with length i at time t, then the probability of finding such a spacing at time t is $P_i(t) = N_i(t)/N$. Let Q_i denote the probability that an arbitrary vehicle moves i sites during a given time step, if $v_{\max} = M$, then we have:

$$\begin{aligned} Q_0 &= P_0 + fP_1 \\ Q_i &= P_i(1-f) + fP_{i+1} (1 \leqslant i \leqslant M-2) \\ Q_{M-1} &= P_{M-1}(1-f) + f\sum_{k=M}^{\infty} P_k \\ Q_M &= (1-f)\sum_{k=M}^{\infty} P_k. \end{aligned} \tag{4}$$

To obtain the P_i, we introduce $W_{i \to j}$ to denote the probability of finding an inter-car spacing with length i at a given time t which changes into length j at the next time $t+1$. According to the Eqs. (1)–(4), one can write all the nonzero $W_{i \to j}$ as follows:

$$\begin{aligned} &W_{0\to 1} = P_0 Q_1, \ldots, W_{0\to M} = P_0 Q_M; \\ &W_{1\to 0} = P_1(1-f)Q_0, \quad W_{1\to 2} = P_1[fQ_1 + (1-f)Q_2] \\ &\qquad \cdots \\ &W_{1\to M} = P_1[fQ_{M-1} + (1-f)Q_{2M}], \quad W_{1\to M+1} = P_1 f Q_M; \end{aligned}$$

$$
\begin{aligned}
&W_{2\to 0} = P_2(1-f)Q_0, \quad W_{2\to 1} = P_2[fQ_0 + (1-f)Q_1] \\
&\qquad\qquad \cdots \\
&W_{2\to M} = P_2[fQ_{M-1} + (1-f)Q_M], \quad W_{2\to M+1} = P_2 fQ_M; \\
&\qquad\qquad \ldots; \\
&W_{M-1\to 0} = P_{M-1}(1-f)Q_0, \quad W_{M-1\to 1} = P_{M-1}[fQ_0 + (1-f)Q_1] \\
&\qquad\qquad \cdots \\
&W_{M-1\to M} = P_{M-1}[fQ_{M-1} + (1-f)Q_M], \quad W_{M-1\to M+1} = P_{M-1} fQ_M; \\
&\qquad\qquad n >= M: \\
&W_{n\to n-M} = P_n(1-f)Q_0, \quad W_{n\to n-(M-1)} = P_n[(1-f)Q_1 + fQ_0] \\
&\qquad\qquad \cdots \\
&W_{n\to n-1} = P_n[(1-f)Q_{M-1} + fQ_{M-2}], \quad W_{n\to n+1} = P_2 fQ_M;
\end{aligned} \tag{5}
$$

i.e.

$$
\begin{aligned}
&W_{0\to j} = P_0 Q_j \quad (1 \leqslant j \leqslant M) \\
&W_{i\to 0} = P_i(1-f)Q_0 \quad (1 \leqslant i \leqslant M-1) \\
&W_{i\to j} = P_i[fQ_{j-1} + (1-f)Q_j] \quad (1 \leqslant i \leqslant M-1,\ 1 \leqslant j \leqslant M,\ i \neq j) \\
&W_{i\to M+1} = P_i fQ_M \quad (1 \leqslant i \leqslant M-1) \\
&W_{i\to i-M} = P_i(1-f)Q_0 \quad (i \geqslant M) \\
&W_{i\to i-(M-j)} = P_i[fQ_{j-1} + (1-f)Q_j] \quad (i \geqslant M,\ 1 \leqslant j \leqslant M-1) \\
&W_{i\to i+1} = P_i fQ_M \quad (i \geqslant M) \\
&W_{i\to j} = 0, \quad \text{otherwise.}
\end{aligned} \tag{6}
$$

When the system approaches its asymptotic steady state, all the P_i will cease to change; thus the steady condition for the steady state holds:

$$\sum_{i\neq m} W_{i\to m} = \sum_{i\neq m} W_{m\to i}, \quad \forall m. \tag{7}$$

i.e.

$$
\begin{aligned}
&W_{1\to 0} + W_{2\to 0} + \cdots + W_{M\to 0} = W_{0\to 1} + W_{0\to 2} + \cdots + W_{0\to m}, (m = 0) \\
&\qquad\qquad \cdots\cdots \\
&W_{m\to m-M} + W_{m\to m-(M-1)} + \cdots + W_{m\to m+1} \\
&\quad = W_{m+M\to m} + W_{m+M-1\to m} + \cdots + W_{m-1\to M}, \quad (m > M+1)
\end{aligned} \tag{8}
$$

Above all, we get a hierarchy of non-closed equations since the equation for the P_i depends on the P_j, $(j > i)$. Although it seems as could not be solved, it has finite nonzero P_i for the physical restrict. In the model if L and N is fixed, one can obtain the maximum empty sites $i_{\max}$ immediately. This maximum value is $L-N$, i.e. $i_{\max} = L-N$. Then $P_i = 0$ when $i > L-N$, since the maximum inter-car spacing must be less than $L-N$. There are $L-N+1$ variables $P_0, P_1, \ldots, P_{L-N}$ in Eqs. (8), which satisfy the following two equations:

$$\sum_{k=0}^{L-N} P_k = 1, \quad \sum_{k=1}^{L-N} kP_k = \frac{1}{\rho} - 1 \tag{9}$$

we should truncate Eq. (8) and obtain other $L-N-1$ equations. In the present paper, we will choose the $L-N-1$ equations with m from 0 to $L-N-2$. Thus, we will have $L-N+1$ independent equations, from which one can readily work out the values of $P_0, P_1, \ldots, P_{L-N}$, then the average speed of the traffic in the steady state is:

$$
\begin{aligned}
\langle v(t\to\infty)\rangle &= \sum_{i=1}^{M} P_i[(i-1)f + i(1-f)] + \sum_{i=M+1}^{L-N} P_i[(M-1)f + M(1-f)] \\
&= \sum_{i=1}^{M} iP_i + \sum_{i=M+1}^{L-N} MP_i - f(1-P_0)
\end{aligned} \tag{10}
$$

Based on the above discussions, one can easily obtain the traffic flux of the steady state:

$$J(t\to\infty) = \rho\langle v(t\to\infty)\rangle. \tag{11}$$

3. Numerical simulations in comparison with theoretical results

In order to compare with the analytical results, we performed numerical simulations on the model with length $L = 1000$ and the maximum car velocity $M = 2$. The number of vehicles N is adjusted so as to give the desired vehicle density ρ. The first 5000 time steps are excluded from the averaging procedure so as to remove the transient behavior. The average values are taken over the next 1000 time steps. For the case $M = 2$:

$$\begin{aligned}
&W_{0\to1} = P_0Q_1, \quad W_{0\to2} = P_0Q_2;\\
&W_{1\to0} = P_1(1-f)Q_0, \quad W_{1\to2} = P_1[fQ_1 + (1-f)Q_2, \quad W_{1\to3} = P_1fQ_2;\\
&W_{n\to n-2} = P_n(1-f)Q_0, \quad W_{n\to n-1} = P_n[(1-f)Q_1 + fQ_0, \quad W_{n\to n+1} = P_nfQ_2, \quad (2 \leqslant n \leqslant L - L\rho);\\
&W_{i\to j} = 0, \quad \text{otherwise.}
\end{aligned} \tag{12}$$

For the steady state Eq. (12) holds, i.e.:

$$\begin{aligned}
&(Q_1 + Q_2)P_0 = (1-f)Q_0P_1 + (1-f)Q_0P_2, (m = 0);\\
&P_1[(1-f)Q_0 + fQ_1 + Q_2] = Q_1P_0 + [(1-f)Q_1 + fQ_0]P_2 + (1-f)Q_0P_3, (m = 1);\\
&P_2[Q_0 + (1-f)Q_1 + fQ_2] = Q_2P_0 + P_1[(1-f)Q_2 + fQ_1] + [(1-f)Q_1 + fQ_0]P_3 + (1-f)Q_0P_4, (m = 2);\\
&P_3[Q_0 + (1-f)Q_1 + fQ_2] = fP_1Q_2 + fQ_2P_2 + P_4[(1-f)Q_1 + fQ_0] + (1-f)Q_0P_5, (m = 3);
\end{aligned}$$

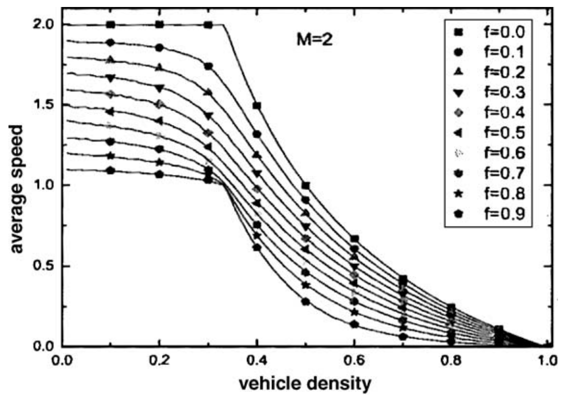

Fig. 1. The fundamental diagram of the average speed with the maximum speed $M = 2$ and for different stochastic delay probabilities f. The solid curves are numerical simulations while the points with different symbols represent theoretical results. The curves from the top down along the vehicle velocity axis correspond to different values of f ranging from 0 to 0.9, in steps of 0.1.

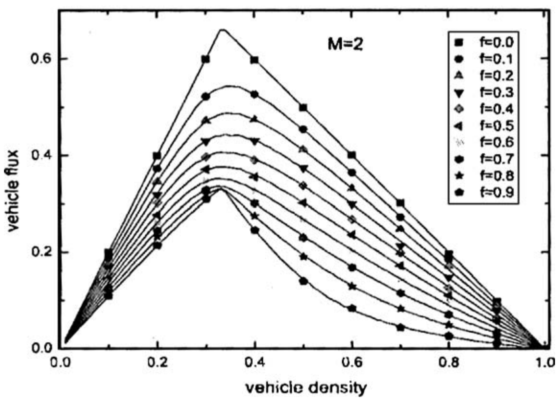

Fig. 2. The fundamental diagram of the vehicle flux with the maximum speed $M = 2$ and for different stochastic delay probabilities f. The solid curves are numerical simulations while the points with different symbols represent theoretical results. The curves from the top down along the traffic flux axis correspond to different values of f ranging from 0 to 0.9, in steps of 0.1.

$$\begin{aligned} P_m[Q_0 + (1-f)Q_1 + fQ_2] &= fP_{m-1}Q_2 + [(1-f)Q_1 + fQ_0]P_{m+1} + (1-f)Q_0P_{m+2}, (m=4). \\ &\cdots\cdots \\ P_m[Q_0 + (1-f)Q_1 + fQ_2] &= fP_{m-1}Q_2 + [(1-f)Q_1 + fQ_0]P_{m+1} + (1-f)Q_0P_{m+2}, (m=L-L\rho-2). \end{aligned} \tag{13}$$

Then for a given density ρ, we can truncate Eq. (12) and obtain $L - L\rho - 1$ equations, i.e. Eq. (13) (e.g. if $\rho = 0.1, 0.2, \ldots$ the corresponded $(L - L\rho - 1) = 0.9L - 1, 0.8L - 1, \ldots)$ and with the Eq. (9) we can work out the values of $P_i, i = 0, 1, \ldots, L - N$. Finally, we get the average speed and the flux of the traffic in the steady state. Figs. 1 and 2 show comparisons between results obtained from numerical simulations and theory over the entire range of the vehicle density ρ. The curves are the simulation results, while the symbols represent the theoretical results. Theoretical results are in excellent agreement with numerical simulations.

4. Discussion

In summary, we propose and study a one-dimensional CA model of high-speed vehicles with the FI acceleration rule and the NS stochastic delay mechanism. The analysis of the dynamical evolution of our model give us a clearly physical picture of how the acceleration and stochastic delay rules affect the evolution and the corresponding asymptotic steady state. Although cellular automata are designed for efficient computer simulation studies, an analytical description is possible, though difficult all too often. The method presented here can also be applied to other CA models. Our study may shed some new light on developing analytical approaches to other traffic models. The question, how the stationary state is approached, is still an important open issue, which is currently under investigation and promises to yield new insights into the physics of the NS model.

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