Phase synchronization on scale-free networks with community structure

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Abstract

In this Letter, we propose a growing network model that can generate scale-free networks with a tunable community strength. The community strength, $C$, is directly measured by the ratio of the number of external edges to that of the internal ones; a smaller $C$ corresponds to a stronger community structure. By using the Kuramoto model, we investigated the phase synchronization on this network and found an abnormal region ($C \leq 0.002$), in which the network has even worse synchronizability than the unconnected case ($C = 0$). On the other hand, the community effect will vanish when $C$ exceeds 0.1. Between these two extreme regions, a stronger community structure will hinder global synchronization.

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Synchronization is observed in many natural, social, physical and biological systems, and has found applications in a variety of fields [1]. The large number of coupled dynamical systems that exhibit synchronized states are subjects of great interest. In the early stage, these studies are restricted to either the regular networks [2,3], or the random ones [4,5]. Recently, inspired by the new discovery of several common characteristics of real networks [6–9], the majority of the studies about network synchronization focus on networks with complex topologies. The effects of average distance [10–12], heterogeneity [13–16], clustering [17,18], and weight distribution [19–22] on network synchronizability have been extensively investigated.

Besides the small-world [23] and scale-free [24] properties, it has been demonstrated that many real networks have the so-called community structure [25,26]. Qualitatively, a community is defined as a subset of nodes within a network such that connections among the nodes therein are denser than that with the rest of the network [27]. Very recently, by applying the epidemiological models on community networks, it was found that the network epidemic dynamics are highly affected by the community structure [28,29]. To date, however, the issue of synchronization on community networks has not been fully investigated [30,31]. Based on a toy network model with a tunable community strength, in this Letter we intend to provide a first analysis on how community structure affects the network phase synchronization.

Our model starts from $n$ community cores; each core contains $m_0$ fully connected nodes. Initially, there are no connections among different community cores. At each time step, to each community core, one node is added. Thus, there are in total $n$ new nodes being added in one time step. Each node will attach $m$ edges to existing nodes within the same community, and simultaneously $m'$ edges to existing nodes outside this community core. The former are internal edges, and the latter are external edges. Note that, the $m$ and $m'$ are not necessary to be integers. For example, to generate 2.7 edges can be implemented as follows: Firstly, generate 2 edges, and then generate the third one with probability 0.7. Similar to the evolutionary mechanism of Barabási–Albert (BA) networks [24], we

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assume the probability of choosing an existing node $i$ to connect to is proportional to $i$’s degree $k_i$. Each community core will finally become a single community of size $N_c$, and the network size $N = nN_c$. By using the rate-equation approach [32], one can easily obtain the degree distribution of the whole network, $p(k) \sim k^{-3}$. For simplicity, we directly use the ratio of external edges to that of the internal ones, $C = m'/m$, to measure the strength of the community structure. Clearly, a smaller $C$ corresponds to sparser external edges thus a stronger community structure.

In contrast to the analysis methods of master stability function [33–37], hereinafter, we investigate the nonidentical oscillators based on the Kuramoto model [38–40], which is described by the coupled differential equations

$$\frac{d\phi_i}{dt} = \omega_i - \sigma \sum_j a_{ij} \sin(\phi_i - \phi_j),$$

where $0 \leq \phi_i < 2\pi$ are phase variables, $\omega_i$ are intrinsic frequencies, $i = 1, 2, \ldots, N$, $\sigma$ is the coupling strength, and $[a_{ij}]$ is the adjacency matrix ($a_{ij} = 1$ iff nodes $i$ and $j$ are connected). Initially, $\phi_i$ and $\omega_i$ are randomly and uniformly distributed in the intervals $[0, 2\pi)$ and $[-0.5, 0.5]$, respectively. Numerical results are obtained by integrating Eqs. (1) using the Runge–Kutta method with step size 0.01. To characterize the synchronized states, we use the order parameter

$$M = \left\{ \left. \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j} \right| \right\},$$

where $\{\}$ signifies the time averaging. The order parameters are averaged over $10^4$ time steps, excluding the former 5000 time steps, to allow for relaxation to a steady state. Clearly, $M$ is of order $1/\sqrt{N}$ if the oscillators are completely uncoupled ($\sigma = 0$), and will approach 1 if they are all in the same phase.

Fig. 1 reports the simulation results for different community strength $C$. Whatever the value of $C$, the parameter $M$ increases sharply after a critical point $\sigma_c = 0.6$. This point is just the same as the critical point at which synchronized behavior emerges in each separate community (each community is a BA network of size 1000 and average degree 6). For all the cases with $C \geq 0.003$, a strong community structure (i.e. a smaller $C$) hinders global synchronization. It is due to the difficulty to harmonize different communities based only on a very few external edges. The community effect becomes lower as $C$ increases. For sufficiently large $C$ (see the cases of $C = 0.15$ and $C = 0.30$ in Fig. 1), the network synchronized behavior is almost the same as that of the original BA network; that is, community effect vanishes. Consider a network consisting of $n$ unconnected communities ($C = 0$), in which each community itself can approach a nearly completely synchronized state. The order parameter $M$ of the whole network is

$$M(n) = \frac{1}{n(2\pi)^n} \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \cdots \int_0^{2\pi} d\phi_n \chi,$$

where $\chi = \sqrt{(\sum_i \sin \phi_i)^2 + (\sum_i \cos \phi_i)^2}$. The numerical result $M(5) \approx 0.403$ is represented by a horizontal line in Fig. 1. The corresponding simulation with $C = 0$ shows that $M$ slightly fluctuates around 0.403 for sufficiently large $\sigma$, which is in accordance with the above numerical result. Interestingly, we found an abnormal region $C \leq 0.002$, in which the networks have even worse synchronizability than the unconnected case ($C = 0$). Within this region, a community cannot get harmonized with other communities through its few external edges. On the contrary, the input signals contained by these external edges disturb the internal synchronizing process of this community. Fig. 2 shows the order parameter $M$ vs $C$ for different $\sigma$. As mentioned above, when $C \leq 0.002$, $M$ cannot reach the dash line ($M = 0.403$). We have also checked that even for very large $\sigma$, $M$ is always smaller than 0.403 if $C \leq 0.002$. The distinct difference between the original BA networks and the present community networks vanishes when the density of external edges exceeds 0.1. The division of those three regions is qualitative, and the borderlines between neighboring regions.
cannot be exactly determined. However, it provides a clearer picture about the effect of the community structure.

To further understand the underlying mechanism of synchronization on community networks, we investigate the partial synchronization within a separate community. For the $i$th community, the corresponding order parameter $M_i$ is defined as

$$M_i = \left\{ \frac{1}{N_c} \sum_j e^{i\phi_j} \right\},$$  

where the sum goes over all the nodes belonging to the $i$th community. Fig. 3 exhibits the temporal behaviors of order parameters for the whole network and for each community. The four plots correspond to the cases of $C = 0.001$, $C = 0.003$, $C = 0.02$, and $C = 0.15$, respectively. In the abnormal region ($C = 0.001$), due to the external disturbance, the order parameter of each community is remarkably below 1 even in the long time limit. After $M_i$ reaches its steady value ($t \approx 4$), $M$ also becomes steady, indicating that the external edges cannot harmonize different communities, but only introduce some noise thus hinder the partial synchronization. When $C \geq 0.003$, each community itself can approach a nearly completely synchronized state, and can harmonize with other communities in some extent; therefore, $M$ will continuously increase after $M_i$ gets steady. For sufficiently large $C$ (see the case of $C = 0.15$), the whole network can approach the nearly completely synchronized state almost as quickly as the single separate community, and the effect of the community structure on network dynamics is hardly observed (at least for the Kuramoto model).

In conclusion, we have proposed a scale-free network model with a tunable community strength and studied its synchronization phenomenon. We have investigated the Kuramoto model in community networks and found an abnormal region, in which the networks have even worse synchronizability than the unconnected case. Due to the complicity of the scale-free structure itself, we are unable to give a theoretical and analytical explanation about this observed phenomenon. An approximate analytic solution of a similar phenomenon may be obtained based on a more ideal community network model, where each community is a complete graph (very recently, this ideal model has been investigated, showing the effect of the modular number on network synchronizability [31]). Beyond this abnormal region, analogous to the result from the approach of the master stability function, increasing the density of external edges will sharply enhance the network synchronizability. However, when the density of external edges exceeds 0.1, the synchronized behavior becomes almost the same as that of the original BA networks and further enhancement cannot be achieved. This result is not only of theoretical interest, but also significant in practice if one wants to enhance the synchronizability of a community network by adding external edges. Finally, we would like to point out that, although the present model is very simple, it provides a useful framework to have detailed understanding about the effect of the community structure on network dynamics since the community strength in the model is adjustable. We hope this model can also be applied to the studies on many other network dynamical processes.

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